International Journal of Theoretical Physics, Vol. 44, No. 9, September 2005 (© 2005)
DOI: 10.1007/s10773-005-4835-2

General Relativity and Cosmology Derived From Principle of Maximum Power or Force

Christoph Schiller[1]

Received March 14, 2005; accepted March 31, 2005

The field equations of general relativity are shown to derive from a limit to force or to power in nature. The limits have the value of $c^4/4G$ and $c^5/4G$. The proof makes use of a result of Jacobson. All known experimental data are consistent with the limits. Applied to the universe, the limits predict its darkness at night and the observed scale factor. Other experimental tests of the limits are proposed. The main counterarguments and paradoxes are discussed, such as the transformation under boosts, the force felt at a black hole horizon, the mountain problem, and the contrast to scalar–tensor theories of gravitation. The resolution of the paradoxes also clarifies why the maximum force and the maximum power have remained hidden for so long. The derivation of the field equations shows that the maximum force or power plays the same role for general relativity as the maximum speed plays for special relativity.

KEY WORDS: maximum force; maximum power; general relativity; horizon.

1. INTRODUCTION

A simplification of general relativity has welcome effects on the teaching of the topic. Some years ago, a strong simplification has been reported by Gibbons (Gibbons, 2002) and independently, by the present author (Schiller, 1997–2004) General relativity was shown to derive from the so-called maximum force (or maximum power) principle: *There is a maximum force (and power) in nature:*

$$F \leq \frac{c^4}{4G} = 3.0 \times 10^{43}\,\mathrm{N} \qquad \text{and} \qquad P \leq \frac{c^5}{4G} = 9.1 \times 10^{51}\,\mathrm{W}. \tag{1}$$

Either of the two equivalent expressions can be taken as basic principle. So far, the arguments used for the connection between these limits and general relativity were either quite abstract or rather heuristic. The present paper gives a derivation of the field equations from either of the limit values and shows the equivalence of the two formulations of general relativity. This paper also includes the discussion

[1] Innere Wiener Straße 52, 81667 München, Germany; e-mail: cs@motionmountain.net.

of the main paradoxes, uses the limits (1) to deduce the central points of cosmology and suggests some new experimental tests of general relativity.

The concept of force needs careful use in general relativity. Force is the change of momentum with time. Since momentum is a conserved quantity, force is best visualized as an upper limit for the rate of flow of momentum (through a given physical surface). Only with this clarification does it make sense to use the concept of force in general relativity.

The value $c^4/4G$ of the force limit is the energy of a Schwarzschild black hole divided by twice its radius. The maximum power $c^5/4G$ is realized when such a black hole is radiated away in the time that light takes to travel along a length corresponding to twice the radius. It will become clear below why a Schwarzschild black hole, as an extremal case of general relativity, is necessary to realize these limit values.

Generally speaking, the aim is to prove that the principle of maximum force (or that of maximum power) plays for general relativity the same role that the principle of maximum speed plays for special relativity. This unconventional analogy (Schiller, 1997–2004) requires a proof in several steps. First, one has to *derive* the field equations of general relativity from the maximum force. Then one has to show that *no imaginary* set-up or situation—thus no Gedanken experiment—can overcome the limit. Subsequently, one has to show that *no experimental data* contradicts the statement of maximum force. Finally, one has to deduce predictions for future experimental tests made on the basis of maximum force or power. These are the same steps that have led to the establishment of the idea of a maximum speed of nature. These steps structure this paper.

Since force (respectively, power) is change of momentum (energy) with time, the precise conditions for momentum (energy) measurement must be specified, in a way applicable in a space–time that is curved. Momentum, like energy, is a conserved quantity. Any change of momentum or energy thus happens through flow. As a result, a maximum force (respectively, power) value in nature implies the following statement: one imagines a physical surface and completely covers it with observers; then the integral of all momentum (respectively, energy) values flowing through that surface per unit time, measured by all those observers, never exceeds the maximum value. It plays no role how the surface is chosen, as long as it is *physical*, i.e., as long as the surface allows to fix observers on it.

A condition for such a measurement is implicit in the surface flow visualization. The local momentum (or energy) change for each observer is the value that each observer measures for the flow at precisely his or her position. The same condition of observer proximity is also required for speed measurements in special relativity.

Since 3-force and power appear together in the force 4-vector, both the force and the power limits are equivalent and inseparable. It is sometimes suggested that the theory of general relativity does not admit a concept of force or of its zeroth component, power. This is not correct; the value of force is simply so strongly dependent on observer choices that usually one tends to avoid the concept of force altogether. On the contrary, it turns out that *every* quantity with the dimensions of force (or of power) that is measured by an observer is bound by the limit value. This can either be the magnitude of the four vector or the value of any of its four components. The force limit $c^4/4G$ (and the corresponding power limit) is valid for all these observables, as will become clear below. In particular, it will be shown below why an arbitrary change of coordinates does *not* allow to exceed the force or power limit, contrary to expectation. This result of general relativity is equivalent to the result of special relativity that a change of coordinates does not allow to exceed the speed limit.

The maximum force and power are also given, within a factor 1/4, by the Planck energy divided by the Planck length, respectively, by the Planck time. The origin of the numerical coefficient 1/4 has no deeper meaning. It simply turns out that 1/4 is the value that leads to the correct form of the field equations of general relativity.

2. THE DERIVATION OF GENERAL RELATIVITY

To derive the theory of relativity one has to study those systems that realize the limit value of the observable under scrutiny. In the case of the *special* theory of relativity, the systems that realize the limit speed are light and massless particles. In the *general* theory of relativity, the systems that realize the limit are less obvious. One notes directly that a maximum force (or power) cannot be realized across a *volume* of space. If that were the case, a simple boost could transform the force (or power) to a higher value. Nature avoids this by realizing maximum force and power only on surfaces, not volumes, and at the same time by making such surfaces unattainable. These unattainable surfaces are basic to general relativity; they are called *horizons*. Maximum force and power only appear on horizons (Schiller, 1997–2004). The definition of a horizon as a surface of maximum force (or power) is equivalent to the more usual definition as a surface that provides a limit to signal reception, i.e., a limit to observation.

The reasoning in the following will consist of three additional steps. First, it will be shown that a maximum force or power implies that unattainable surfaces are always curved. Then, it will be shown that any curved horizon follows the so-called horizon equation. Finally, it will be shown that the horizon equation implies general relativity. (In fact, the sequence of arguments can also be taken in the opposite direction; all these steps are equivalent to each other.)

The connection between horizons and the maximum force is the central point in the following. It is as important as the connection between light and the maximum speed in special relativity. In special relativity, one shows that light speed, being the maximum speed in nature, implies the Lorentz transformations. In general relativity, one must show that horizon force, being the maximum force in nature, implies the field equations. To achieve this aim, one starts with the realization that all horizons show energy flow at their location. There is no horizon without energy flow. This connection implies that a horizon cannot be a plane, as an infinitely extended plane would imply an infinite energy flow.

The simplest finite horizon is a static sphere. A spherical horizon is characterized by its curvature radius R or equivalently, by its surface gravity a; the two quantities are related by $2aR = c^2$. The energy flow moving though any horizon is always finite in length, when measured along the propagation direction. One can thus speak more specifically of an energy pulse. Any energy pulse through a horizon is thus characterized by an energy E and a proper length L. When the energy pulse flows perpendicularly through a horizon, the momentum change or force for an observer at the horizon is

$$F = \frac{E}{L}. \tag{2}$$

The goal is to show that maximum force implies general relativity. Now, the maximum force is realized on horizons. One thus needs to insert the maximum possible values for each of these quantities and to show that general relativity follows.

Using the maximum force value and the area $4\pi R^2$ for a spherical horizon one gets

$$\frac{c^4}{4G} = \frac{E}{LA} 4\pi R^2. \tag{3}$$

The fraction E/A is the energy per area flowing through any area A that is part of a horizon. The insertion of the maximum values is complete when one notes that the length L of the energy pulse is limited by the radius R. The limit $L \leq R$ is due to geometrical reasons; seen from the concave side of the horizon, the pulse must be shorter than the curvature radius. An independent argument is the following. The length L of an object accelerated by a is limited by special relativity (D'Inverno, 1992; Rindler, 2001) by

$$L \leq \frac{c^2}{2a}. \tag{4}$$

Special relativity already shows that this limit is due and related to the appearance of a horizon. Together with relation (3), the statement that horizons are surfaces of maximum force leads to the following central relation for static, spherical horizons:

$$E = \frac{c^2}{8\pi G}\, a\, A. \tag{5}$$

This *horizon equation* relates the energy flow E through an area A of a spherical horizon with surface gravity a. The horizon equation follows from the idea that horizons are surfaces of maximum force. The equation states that the energy flowing through a horizon is limited, that this energy is proportional to the area of the horizon, and that the energy flow is proportional to the surface gravity.

The above derivation also yields the intermediate result

$$E \leq \frac{c^4}{16\pi G} \frac{A}{L}. \tag{6}$$

This form of the horizon equation states more clearly that no surface other than a horizon can reach the limit energy flow, given the same area and pulse length (or surface gravity). No other part of physics makes comparable statements; they are an intrinsic part of the theory of gravitation.

Another variation of the derivation of the horizon starts with the emphasis on power instead of on force. Using $P = E/T$ as starting equation, changing the derivation accordingly, also leads to the horizon equation.

It is essential to stress that the horizon equations (5) or (6) follow from only two assumptions: first, there is a maximum speed in nature, and second, there is a maximum force (or power) in nature. No specific theory of gravitation is assumed. The horizon equation might even be testable experimentally, as argued below. (One also notes that the horizon equation—or, equivalently, the force or power limits—imply a maximum mass change rate in nature given by $dm/dt \leq c^3/4G$.) In particular, up to this point it was *not* assumed that general relativity is valid; equally, it was *not* assumed that spherical horizons yield Schwarzschild black holes (indeed, other theories of gravity also lead to spherical horizons).

Next, one has to generalize the horizon equation from static and spherical horizons to general horizons. Since the maximum force is assumed to be valid for *all* observers, whether inertial or accelerating, the generalization is straightforward. For a horizon that is irregularly curved or time-varying the horizon equation becomes

$$\delta E = \frac{c^2}{8\pi G} a \, \delta A. \tag{7}$$

This differential relation—it might be called the *general horizon equation*—is valid for any horizon. It can be applied separately for every piece δA of a dynamic or spatially changing horizon. The general horizon equation (7) is known to be equivalent to general relativity at least since 1995, when this equivalence was implicitly given by Jacobson (Jacobson, 1995). It will be shown that the differential horizon equation has the same role for general relativity as $dx = c\, dt$ has for special relativity. From now on, when speaking of the horizon equation, the general, differential form (7) of the relation is implied.

It is instructive to restate the behavior of energy pulses of length L in a way that holds for any surface, even one that is not a horizon. Repeating the above derivation, one gets

$$\frac{\delta E}{\delta A} \leq \frac{c^4}{16\pi G} \frac{1}{L}. \tag{8}$$

Equality is only reached in the case that the surface A is a horizon. In other words, whenever the value $\delta E/\delta A$ approaches the right hand side, a horizon is formed. This connection will be essential in the discussion of apparent counterexamples to the limit values.

If one keeps in mind that on a horizon, the pulse length L obeys $L \leq c^2/2a$, it becomes clear that the general horizon equation is a consequence of the maximum force $c^4/4G$ or the maximum power $c^5/4G$. In addition, the horizon equation takes also into account maximum speed, which is at the origin of the relation $L \leq c^2/2a$. The horizon equation thus follows purely from these two limits of nature. One notes that one can also take the opposite direction of arguments: it is possible to derive the maximum force from the horizon equation (7). The two statements are thus equivalent.

One notes that the differential horizon equation is also known under the name "first law of black hole mechanics" (Wald, 1993). The arguments so far thus show that the first law of black hole mechanics is a consequence of the maximum force or power in nature. This connection does not seem to appear in the literature so far. The more general term "horizon equation" used here instead of "first law" makes three points: first, the relation is valid for any horizon whatsoever; second, horizons are more fundamental and general entities than black holes are; third, horizons are limit situations for physical surfaces.

The remaining part of the argument requires the derivation of the field equations of general relativity from the general horizon equation. The derivation—in fact, the equivalence—was implicitly provided by Jacobson (Jacobson, 1995), and the essential steps are given in the following. (Jacobson did not stress that his derivation is valid also for continuous space–time and that his argument can also be used in classical general relativity.) To see the connection between the general horizon equation (7) and the field equations, one only needs to generalize the general horizon equation to general coordinate systems and to general directions of energy-momentum flow. This is achieved by introducing tensor notation that is adapted to curved space–time.

To generalize the general horizon equation, one introduces the general surface element $d\Sigma$ and the local boost Killing vector field k that generates the horizon (with suitable norm). Jacobson uses the two quantities to rewrite the left hand side of the general horizon equation (7) as

$$\delta E = \int T_{ab} k^a \, d\Sigma^b, \tag{9}$$

where T_{ab} is the energy-momentum tensor. This expression obviously gives the energy at the horizon for arbitrary coordinate systems and arbitrary energy flow directions.

Jacobson's main result is that the the right hand side of the general horizon equation (7) can be rewritten, making use of the (purely geometric) Raychaudhuri equation, as

$$a\ \delta A = c^2 \int R_{ab} k^a\, d\Sigma^b, \tag{10}$$

where R_{ab} is the Ricci tensor describing space–time curvature. This relation thus describes how the local properties of the horizon depend on the local curvature. One notes that the Raychaudhuri equation is a purely geometric equation for manifolds, comparable to the expression that links the curvature radius of a curve to its second and first derivative. In particular, the Raychaudhuri equation does *not* contain any implications for the physics of space–times at all.

Combining these two steps, the general horizon equation (7) becomes

$$\int T_{ab} k^a\, d\Sigma^b = \frac{c^4}{8\pi G} \int R_{ab} k^a\, d\Sigma^b. \tag{11}$$

Jacobson then shows that this equation, together with local conservation of energy (i.e., vanishing divergence of the energy-momentum tensor), can only be satisfied if

$$T_{ab} = \frac{c^4}{8\pi G} \left(R_{ab} - \left(\frac{R}{2} + \Lambda \right) g_{ab} \right), \tag{12}$$

where R is the Ricci scalar and Λ is a constant of integration whose value is not specified by the problem. These are the full field equations of general relativity, including the cosmological constant Λ. The field equations thus follow from the horizon equation. The field equations are therefore shown to be valid at horizons.

Since it is possible, by choosing a suitable coordinate transformation, to position a horizon at any desired space–time event, the field equations must also be valid over the whole of space–time. This conclusion completes the result by Jacobson. Since the field equations follow, via the horizon equation, from maximum force, one has thus shown that at every event in nature the same maximum possible force holds; its value is an invariant and a constant of nature.

The reasoning shown here consisted of four steps. First, it was shown that a maximum force or power implies the existence of unattainable surfaces, which were called horizons. Second, it was shown that a maximum force or power implies that unattainable surfaces are always curved. Third, it was shown that any curved horizon follows the horizon equation. Fourth, it was shown (in the way done by Jacobson) that the horizon equation implies general relativity.

In other words, the field equations of general relativity are a direct consequence of the limited energy flow at horizons, which in turn is due to the existence of a maximum force (or power). In fact, the argument also works in the opposite direction, since all intermediate steps are equivalences. This includes jacobson's connection between the horizon equation and the field equations of general relativity. Maximum force (or power), the horizon equation, and general relativity are thus *equivalent*. As a result, one finds the corollary that *general relativity implies a maximum force*.

The maximum force (or power) has thus the same double role in general relativity that the maximum speed has in special relativity. In special relativity, the speed of light is the maximum speed; at the same time it is the proportionality constant that connects space and time, as in $dx = c\,dt$. In general relativity, the horizon force is the maximum force; at the same time the maximum force appears (adorned with a factor 2π) in the field equations as the proportionality constant connecting energy and curvature. If one prefers, the maximum force thus describes the elasticity of space–time and at the same time it describes—if one dares to use the simple image of space–time as a medium—the maximum tension to which space–time can be subjected. This double role of material constants both as proportionality factor and as limit value is well-known in material science.

The analogy between special and general relativity can be carried further. In special relativity, maximum speed implies $dx = c\,dt$ and the observation that time changes with observer change. In general relativity, maximum force (or power) imply the horizon equation $\delta E = a\,\delta A\,c^2/8\pi G$ and the observation that space–time is curved. Curvature is a result of the maximum force or maximum power. Indeed, the derivation above showed that a finite maximum force implies horizons that are curved; the curvature of horizons imply the curvature of surrounding space–time.

One might ask whether rotating or charged black holes change the argument that lead to the derivation of general relativity. However, the derivation using the Raychaudhuri equation does not change. In fact, the only change of the argument appears with the inclusion of torsion, which changes the Raychaudhuri equation itself. As long as torsion plays no role, the derivation given above remains valid.

Another question is how the above proof relates to scalar–tensor theories of gravity. If a particular scalar–tensor theory would obey the general horizon equation (7) then it would also show a maximum force. The general horizon equation must be obeyed both for *static* and for *dynamic* horizons. If that is the case, the specific scalar–tensor theory would be equivalent to general relativity, as it would allow, using the argument of Jacobson, to deduce the usual field equations. This case can appear if the scalar field behaves like matter, i.e., if it has mass–energy like matter and curves space–time like matter. On the other hand, if in the particular scalar–tensor theory the general horizon equation (7) is not obeyed for *all moving* horizons—which is the general case, as scalar–tensor theories have more defining constants than general relativity—then the maximum force does not appear and the theory is not equivalent to general relativity. This connection also shows that an experimental test of the horizon equation for *static* horizons only is not sufficient to confirm general relativity; such a test rules out only some, but not all scalar–tensor theories.

3. APPARENT COUNTERARGUMENTS AND PARADOXES

Despite the preceding and other proofs (Gibbons, 2002) for the equivalence of maximum force and the equations of general relativity, the idea of a maximum force is not yet common. Indeed, maximum force, maximum power, and maximum mass change directly induce counterarguments and attempts to exceed the limit.

3.1. The Mountain Attempt

It is possible to define a surface that is so strangely bent that it passes *just below* every nucleus of every atom of a mountain, like the surface A in Fig. 1. All atoms of the mountain above sea level are then *just above* the surface, barely touching it. In addition, one imagines that this surface is moving *upwards* with almost the speed of light. It is not difficult to show that the mass flow through this

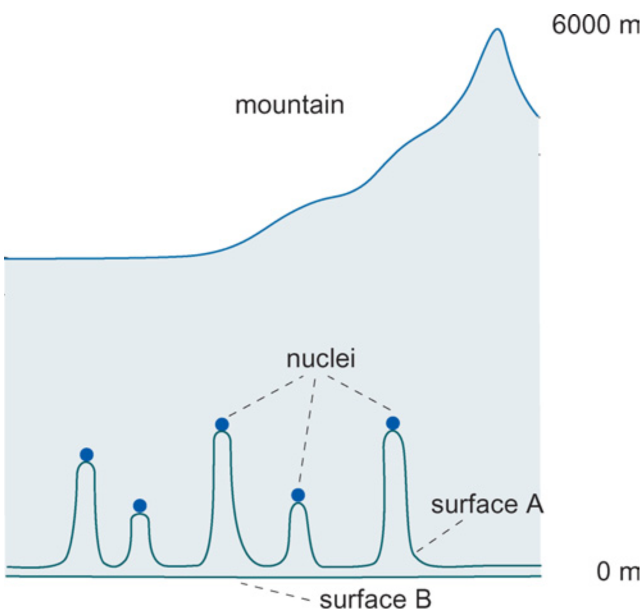

Fig. 1. The mountain problem.

surface is higher than the mass flow limit. Indeed, the mass flow limit $c^3/4G$ has a value of about 10^{35} kg/s; in a time of 10^{-22} s, the diameter of a nucleus divided by the speed of light, only 10^{13} kg need to flow through the surface; that is the mass of a mountain.

The mentioned surface seems to provide a counterexample to the limit. However, a closer look shows that this is not the case. The issue is the expression “just below.” Nuclei are quantum particles and have an indeterminacy in their position; this indeterminacy is essentially the nucleus–nucleus distance. As a result, in order to be sure that the surface of interest has all atoms *above* it, the shape cannot be that of surface A in Fig. 1. It must be a flat plane that remains below the whole mountain, like surface B in the figure. However, a flat surface rising through a mountain does not allow to exceed the mass change limit.

3.2. The Multiple Atom Attempt

One can imagine a number of atoms equal to the number of the atoms of a mountain, but arranged in a way that all lie with large spacing (roughly) in a single plane. Again, the plane could move with high speed. However, also in this case the uncertainty in the atomic positions makes it impossible to say that the mass flow limit has been exceeded.

3.3. The Multiple Black Hole Attempt

Black holes are typically large and their uncertainty in position is thus negligible. The mass limit $c^3/4G$ or power limit $c^5/4G$ correspond to the flow of a single black hole moving through a plane at the speed of light. Several black holes crossing a plane together at just under the speed of light thus seem to beat the limit. However, the surface has to be physical: an observer must be possible at each one of its points. But no observer can cross a black hole. A black hole thus effectively punctures the moving plane surface; no black hole can ever be said to cross a plane surface, even less so a multiplicity of black holes. The limit remains valid.

3.4. The Multiple Neutron Star Attempt

The mass limit seems in reach when several neutron stars (which are slightly less dense than a black hole of the same mass) cross a plane surface at the same time, at high speed. However, when the speed approaches the speed of light, the crossing time for points far from the neutron stars and for those that actually cross the stars differ by large amounts. Neutron stars that are almost black holes cannot be crossed in a short time in units of a coordinate clock that is located far from the stars. Again, the limit is not exceeded.

3.5. The Boost Attempt

A boost can apparently be chosen in such a way that a force value F in one frame is transformed into any desired value F' in the other frame. However, this result is not physical. To be more concrete, one imagines a massive observer, measuring the value F, at rest with respect to a large mass, and a second, primed observer moving toward the large mass with relativistic speed, measuring the value F'. Both observers can be thought to be as small as desired. If one transforms the force field at rest F applying the Lorentz transformations, the force F' for the moving observer can apparently reach extremely high values, as long as the speed is high enough. However, a force value must be measured by an observer at the specific point. One has thus to check what happens when the rapid observer moves toward the region where the force would exceed the force limit. The primed observer has a mass m and a radius r. To be an observer, he must be larger than a black hole; in other words, its radius must obey $r > 2Gm/c^2$, implying that the observer has a nonvanishing size. When the observer dives into the force field surrounding the sphere, there will be an energy flow E toward the observer given by the transformed field value and the proper crossing area of the observer. This interaction energy can be made as small as desired, by choosing an observer as small as desired, but it is never zero. When the moving observer approaches the large massive charge, the interaction energy increases continuously. Whatever choice for the smallness of the observer is made is not important. Before the primed observer arrives at the point where the force F' was supposed to be much higher than the force limit, the interaction energy will reach the horizon limits (7) or (8). Therefore, a horizon appears and the moving observer is prevented from observing anything at all, in particular any value above the horizon force.

The same limitation appears when a charged observer tries to measure electromagnetic forces, or when nuclear forces are measured. In summary, boosts do not help to beat the force limit.

3.6. The Divergence Argument

In apparent contrast to what was said so far, the force on a test mass m at a radial distance d from a Schwarzschild black hole (for $\Lambda = 0$) is given by (Ohanian and Ruffini, 1994)

$$F = \frac{GMm}{d^2\sqrt{1 - \frac{2GM}{dc^2}}}. \tag{13}$$

In addition, the inverse square law of universal gravitation states that the force between two masses m and M is

$$F = \frac{GMm}{d^2}. \tag{14}$$

Both expressions can take any value and suggest that no maximum force limit exists.

A detailed investigation shows that the maximum force still holds. Indeed, the force in the two situations diverges only for not physical point-like masses. However, the maximum force implies a minimum approach distance to a mass m given by

$$d_{\mathrm{min}} = \frac{2Gm}{c^2}. \tag{15}$$

The minimum approach distance—simplifying, this would be the corresponding black hole radius—makes it impossible to achieve zero distance between two masses or between a horizon and a mass. The finiteness of this length value expresses that a mass can never be point-like, and that a (real) minimum approach distance of $2Gm/c^2$ ap pears in nature, proportional to the mass. If this minimum approach distance is introduced in Eqs. (13) and (14), one gets

$$F = \frac{c^4}{4G} \frac{Mm}{(M+m)^2} \frac{1}{\sqrt{1 - M/(M+m)}} \leq \frac{c^4}{4G} \tag{16}$$

and

$$F = \frac{c^4}{4G} \frac{Mm}{(M+m)^2} \leq \frac{c^4}{4G}. \tag{17}$$

The maximum force value is never exceeded. Taking this into account the size of observers prevents exceeding the maximum force.

3.7. The Wall Attempt

Force is momentum change. For example, momentum changes when a basketball is reflected from a large wall. If many such balls are reflected at the same time, it seems that a force on the wall larger than the limit can be realized. However, this is impossible. Every wall has a tiny surface gravity. For a large, but finite number of balls, the energy flow limit of the horizon equation (8) will be reached, thus implying the appearance of a horizon. In that case, no reflection is possible anymore, and again the force or power limit cannot be exceeded.

3.8. The Classical Radiation Attempt

It is also not possible to create a force larger than the maximum force concentrating a large amount of light onto a surface. However, the same situation as for basketballs arises: when the limit value E/A given by the horizon equation (8) is reached, a horizon appears that prevents breaking the limit.

3.9. The Multiple Lamp Attempt

It might seem possible to create a power larger than the maximum power by combining two radiation sources that each emit 3/4 of the maximum value. But also in this case, the horizon limit (8) is achieved and thus a horizon appears that swallows the light and prevents that the force or power limit is exceeded. (The limited lifetime of such lamps makes these horizons time-dependent.)

3.10. The Electrical Charge Attempt

One might try to get forces above the limit by combining gravity and electromagnetism. However, in this case, the energy in the horizon equation, like the first law of black hole mechanics (Wald, 1993), only gets an additional term. The energy is then a sum of mass–energy and electromagnetic energy. For example, in the simplest case, that of a static and charged black hole, the energy $\delta E = c^2 \delta m + V \delta q$ crossing the horizon includes the product of the electrical potential V at the horizon and the amount of charge q crossing the horizon. However, the maximum force and power values remain unchanged. In other words, electromagnetism cannot be used to exceed the force or power limit.

3.11. The Inconsistency Objection

If observers cannot be point-like, one might question whether it is still correct to apply the original definition of momentum change or energy change as the integral of values measured by observers attached to a given surface. In general relativity, observers cannot be point-like, as seen above. However, observers can be as small as desired. The original definition thus remains applicable when taken as a limit procedure for an observer size that decreases toward zero. Obviously, if quantum theory is taken into account, this limit procedure comes to an end at the Planck length. This is not an issue for general relativity, as long as the typical dimensions in the situation are much larger than this value.

3.12. The Quantum Attempt

If quantum effects are neglected, it is possible to construct surfaces with sharp angles or even fractal shapes that overcome the force limit. However, such surfaces are not physical, as they assume that lengths smaller than the Planck length can be realized or measured. The condition that a surface be physical implies among others that it has an intrinsic uncertainty given by the Planck length. A detailed study shows that quantum effects do not allow to exceed the horizon force. The basic reason is the mentioned equality of the maximum force with the quotient of the Planck energy and the Planck length, both corrected by a factor of order 1.

Since both the Plan energy and the Planck length are limits in nature, quantum effects do not help at overcoming the force or power limit (Schiller, 1997–2004).

3.13. The Teachings of the Paradoxes

Similar results are found when any other Gedanken experiment is imagined. Discussing them is an interesting way to explore general relativity. No Gedanken experiment is successful; in all cases, horizons prevent that the maximum force is exceeded. Observing a value larger than the force or power limit requires observation across a horizon. This is impossible.

Maximum force (or power) implies that point masses do not exist. This connection is essential to general relativity. The habit of thinking with point masses—a remainder of Galilean physics—is one of the two reasons that the maximum force principle has remained hidden for more than 80 years. The (incorrect) habit of believing that the proper size of a system can be made as small as desired while keeping its mass constant avoids that the force or power limit is noticed. Many paradoxes around maximum force or power are due to this incorrect habit.

To see the use of a maximum force or power for the exploration of gravity, one can use a simple image. Nature prevents large force values by the appearance of horizons. This statement can be translated in engineer's language. To produce a force or power requires an engine. Every engine produces exhausts. When the engine approaches the power limit, the mass of the exhausts is necessarily so large that their gravity cannot be neglected. The gravity of the exhausts saturates the horizon equation and then prevents the engine from reaching the force or power limit.

Force is change of momentum with time; power is change of energy with time. Since both momentum and energy are conserved, all changes take place through a boundary. The force and power limit state that these values are upper limits independently of the boundary that is used. Even if the boundary surface is taken to cross the whole universe, the observed momentum or energy change through that surface is limited by the maximum values. This requires a check with experiments.

4. EXPERIMENTAL DATA

No experiment, whether microscopic—such as particle collisions—macroscopic, or astronomical, has ever measured force values near or even larger than the limit. Also, the search for space–time singularities, which would allow to achieve the force limit, has not been successful. In fact, all force values ever measured are many magnitudes smaller than the maximum value. This result is due to the lack of horizons in the environment of all experiments performed so far.

Similarly, no power measurement has ever provided any exception to the power limit. Only the flow of energy through a horizon should saturate the power limit. Every star, gamma ray burster, supernova, galaxy, or galaxy cluster observed up to now has a luminosity below $c^5/4G$. Also, the energy flow through the night sky horizon is below the limit. (More about this issue below.) The brightness of evaporating black holes in their final phase could approach or equal the limit. So far, none has ever been observed. In the same way, no counterexample to the mass change limit has ever been observed. Finding any counterexample to the maximum force, luminosity, or mass change would have important consequences. It would invalidate the present approach and thus invalidate general relativity.

On the other hand, we have seen above that general relativity contains a maximum force and power, so that every successful test of the field equations underlines the validity of this approach.

The absence of horizons in everyday life is the second reason why the maximum force principle has remained undiscovered for so long. Experiments in everyday life do not point out the force or power limit to explorers. The first reason why the principle remained hidden, as shown above, is the incorrect habit of believing in massive point particles. This is a theoretical reason. (Prejudices against the concept of force in general relativity have also played a role.) The principle of maximum force—or of maximum power—has thus remained unnoticed for a long time because nature hid it both from theorists and from experimentalists.

In short, past experiments do not contradict the limit values and do not require or suggest an alternative theory of gravitation. But neither does the data directly confirm the limits, as horizons are rare in everyday life or in accessible experimental situations. The maximum speed at the basis of special relativity is found almost everywhere; maximum force and maximum power are found almost nowhere. For example, the absence of horizons in particle collisions is the reason that the force limit is not of (direct) importance in this domain.

5. COSMOLOGICAL DATA

A maximum power is the simplest possible explanation of Olbers' paradox. Power and luminosity are two names for the same observable. The sum of all luminosities in the universe is finite; the light and all other energy emitted by all stars, taken together, is finite. If one assumes that the universe is homogeneous and isotropic, the power limit $P \leq c^5/4G$ must be valid across any plane that divides the universe into two halves. The part of the universes's luminosity that arrives on earth is then so small that the sky is dark at night. In fact, the actually measured luminosity is still smaller than this estimate, since a large part of the power is not visible to the human eye (since most of it is matter anyway). In other words, the night is dark because of nature's power limit. This explanation is *not* in contrast to the usual one, which uses the finite lifetime of stars, their finite density, their finite size, the finite age, and the expansion of the universe. In fact, the combination of all these usual arguments simply implies and repeats in more complex words that the maximum power value cannot be exceeded. However, this simple reduction of the traditional explanation seems unknown in the literature.

A maximum force in nature, together with homogeneity and isotropy, implies that the visible universe is of *finite size*. The opposite case would be an infinitely large universe. But in that case, any two halves of the universe would attract each other with a force above the limit (provided the age of the universe is sufficiently large). The result can be made quantitative by imagining a sphere whose center lies at the earth, which encompasses all the universe, and whose radius decreases with time almost as rapidly as the speed of light. The mass flow $dm/dt = \rho A v$ is predicted to saturate the mass flow limit $c^3/4G$; thus one has

$$\frac{dm}{dt} = \rho_o 4\pi R_o^2 c = \frac{c^3}{4G}, \tag{18}$$

a relation also predicted by the Friedmann models. The WMAP measurements confirm that the present day total energy density ρ_o (including dark matter and dark energy) and the horizon radius R_o just saturate the limit value. The maximum force limit thus predicts the observed size of the universe.

In summary, so far, neither experiment nor theory has allowed to exceed the maximum force and power values. Nevertheless, the statement of a maximum force given by $c^4/4G$ (and the corresponding maximum power) remains open to experimental falsification. Since the derivation of general relativity from the maximum force or from the maximum power is now established, one can more aptly call them *horizon* force and *horizon* power.

6. PREDICTIONS

A maximum force and power is equivalent to general relativity and thus implies the inverse square law of gravitation for small speed and curvature values. A maximum force is *not* equivalent to scalar–tensor theories or to modifications of the universal law of gravitation.

The exploration of physical systems that are mathematical analogues of black holes—for example, silent (or acoustical) black holes, or optical black holes—should confirm the force and power limits. Future experiments in these domains might be able to confirm the horizon equations (5) or (7) directly.

Another domain in which tests might be possible is the relation that follows from maximum force for the measurement errors ΔE and Δx. (Schiller, 1997–2004) For all physical systems one has

$$\frac{\Delta E}{\Delta x} \leq \frac{c^4}{4G}. \tag{19}$$

So far, all measurements comply with the relation. In fact, the left side is usually so much smaller than the right side that the relation is not well-known. To have a direct check, one must look for a system where a rough equality is achieved. This might be the case in binary pulsar systems. Other systems do not seem to allow checking the relation. In particular, there does not seem to be a possibility to test this limit in satellite laser ranging experiments. For example, for a position error of 1 mm, the mass error is predicted to be below 3×10^{23} kg, which so far is always the case.

There is a power limit for all energy sources and energies. In particular, the luminosity of all gravitational sources is also limited by $c^5/4G$. Indeed, all formulas for gravitational wave emission contain this value as upper limit (Ohanian and Ruffini, 1994). Similarly, all numerical relativity simulations, such as the power emitted during the merger of two black holes, should never exceed the limit.

The night sky is a horizon. The power limit, when applied to the night sky, makes the testable prediction that the flow of all matter and radiation through the night sky adds up exactly to the value $c^5/4G$. If one adds the flow of photons, baryons, neutrinos, electrons, and the other leptons, including any particles that might be still unknown, the power limit must be precisely saturated. If the limit is exceeded or not saturated, general relativity is not correct. Increasing the precision of this test is a challenge for future investigations.

It might be that one day the amount of matter and energy falling into some black hole, such as the one at the center of the Milky Way, might be measured. If that is the case, the mass rate limit $dm/dt \leq c^3/4G$ could be tested directly.

Perfectly plane waves do not exist in nature. Neither electrodynamic nor gravitational waves can be infinite in extension, as such waves would carry more momentum per time through a plane surface than allowed by the force limit. Taken the other way round, a wave whose integrated intensity approaches the force limit cannot be plane. The power limit thus implies a limit on the product of intensity I (given as energy per time and area) and curvature radius R of the front of a wave moving with the speed of light c:

$$4\pi R^2 I \leq \frac{c^5}{4G}. \tag{20}$$

This statement is difficult to check experimentally, whatever the frequency and type of wave might be, as the value appearing on the right hand side is extremely large. Possibly, future experiments with gravitational wave detectors, X-ray detectors, gamma ray detectors, radio receivers, or particle detectors will allow testing relation (20) with precision. In particular, the nonexistence of plane gravitational waves also excludes the predicted production of singularities in case that two plane waves collide.

Since the maximum force and power limits apply to all horizons, it is impossible to squeeze mass into smaller regions of space than those given by a region completely limited by a horizon. As a result, a body cannot be denser than a (uncharged, nonrotating) black hole of the same mass. Both the force and power limits thus confirm the Penrose inequality. The limits also provide a strong point for the validity of cosmic censorship.

The power limit implies that the highest luminosity is only achieved when systems emit energy at the speed of light. Indeed, the maximum emitted power is only achieved when all matter is radiated away as rapidly as possible: the emitted power $P = Mc^2/(R/v)$ cannot reach the maximum value if the body radius R is larger than a black hole (the densest bodies of a given mass) or the emission speed v is lower than that of light. The sources with highest luminosity must therefore be of maximum density and emit entities without rest mass, such as gravitational waves, electromagnetic waves, or (maybe) gluons. Candidates to achieve the limit are bright astrophysical sources as well as black holes in evaporation or undergoing mergers.

7. OUTLOOK

In summary, the maximum force principle (or the equivalent maximum power principle) was shown to allow a simple axiomatic formulation of general relativity: the horizon force $c^4/4G$ and the horizon power $c^5/4G$ are the highest possible force and power values. General relativity follows from these limits. All known experimental data is consistent with the limits. Moreover, the limits imply the darkness at night and the finiteness of the universe.

It is hoped that the maximum force principle will have applications for the teaching of the field. The principle might bring general relativity to the level of first-year university students; only the concepts of maximum force, horizon, and curvature are necessary.

Apart from suggesting some experimental tests, the principle of maximum force also provides a guide for the search of a unified theory of nature that incorporates general relativity and quantum theory. Any unified theory of nature must state that the value $c^4/4G$ is a maximum force. Within an uncertain numerical factor, this is the case for string theory, where a maximum force appears, the so-called string tension. A maximum force is also predicted by loop quantum gravity. Both string theory and loop quantum gravity thus do predict gravity, as long as the predicted maximum force has the correct value.

REFERENCES

D'Inverno, R. (1992). *Introducing Einstein's Relativity*, Clarendon, Oxford, p. 36.

Gibbons, G. W. (2002). The maximum tension principle in general relativity. *Foundations of Physics* **32**, 1891–1901.

Jacobson, T. (1995). Thermodynamics of spacetime: The Einstein equation of state. *Physical Review Letters* **75**, 1260–1263.

Ohanian, H. C. and Ruffini, R. (1994). *Gravitation and Spacetime*, W. W. Norton, New York.

Rindler, W. (2001). *Relativity, Special, General, and Cosmological*, Oxford University Press, Oxford, p. 70.

Schiller, C. (1997–2004). *Motion Mountain—A Hike Beyond Space and Time Along the Concepts of Modern Physics*, available at (http://www.motionmountain.net), Section 7: Maximum Force— A Simple Principle Encompassing General Relativity.

Schiller, C. (1997–2004). *Motion Mountain—A Hike Beyond Space and Time Along the Concepts of Modern Physics*, available at (http://www.motionmountain.net), Section 36: Maximum Force and Minimum Distance—Physics in Limit Statements.

Wald, R. M. (1993). The first law of black hole mechanics. In *Directions in General Relativity*, B. L. Hu, M. P. Ryan, and C. V. Vishveshwara, Proceedings of the 1993 International Symposium, Maryland: Papers in Honor of Charles Misner, Cambridge University Press, Cambridge.